

# **Deep Reinforcement Learning**

Successor representations

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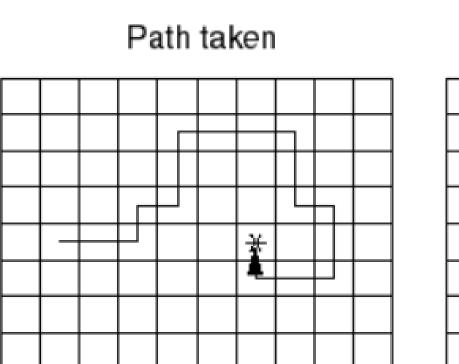
# 1 - Model-based vs. Model-free

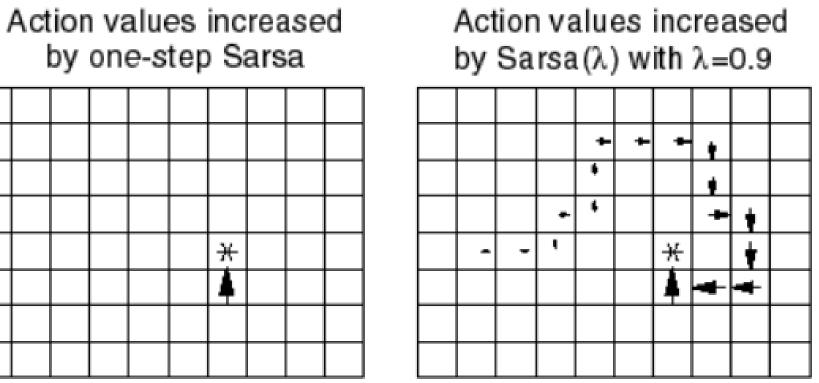
## **Model-based vs. Model-free**

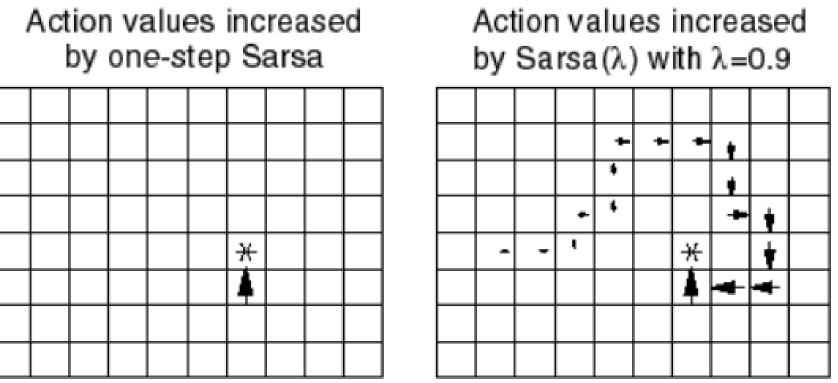
• Model-free methods use the **reward prediction error** (RPE) to update values:

$$egin{aligned} \delta_t &= r_{t+1} + \gamma \, V^\pi(s_{t+1}) - \ & \Delta V^\pi(s_t) &= lpha \, \delta_t \end{aligned}$$

Encountered rewards propagate very slowly to all states and actions.







- If the environment changes (transition probabilities, rewards), they have to relearn everything.
- After training, selecting an action is very fast.

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 $V^{\pi}(s_t)$ 

# **Model-based vs. Model-free**

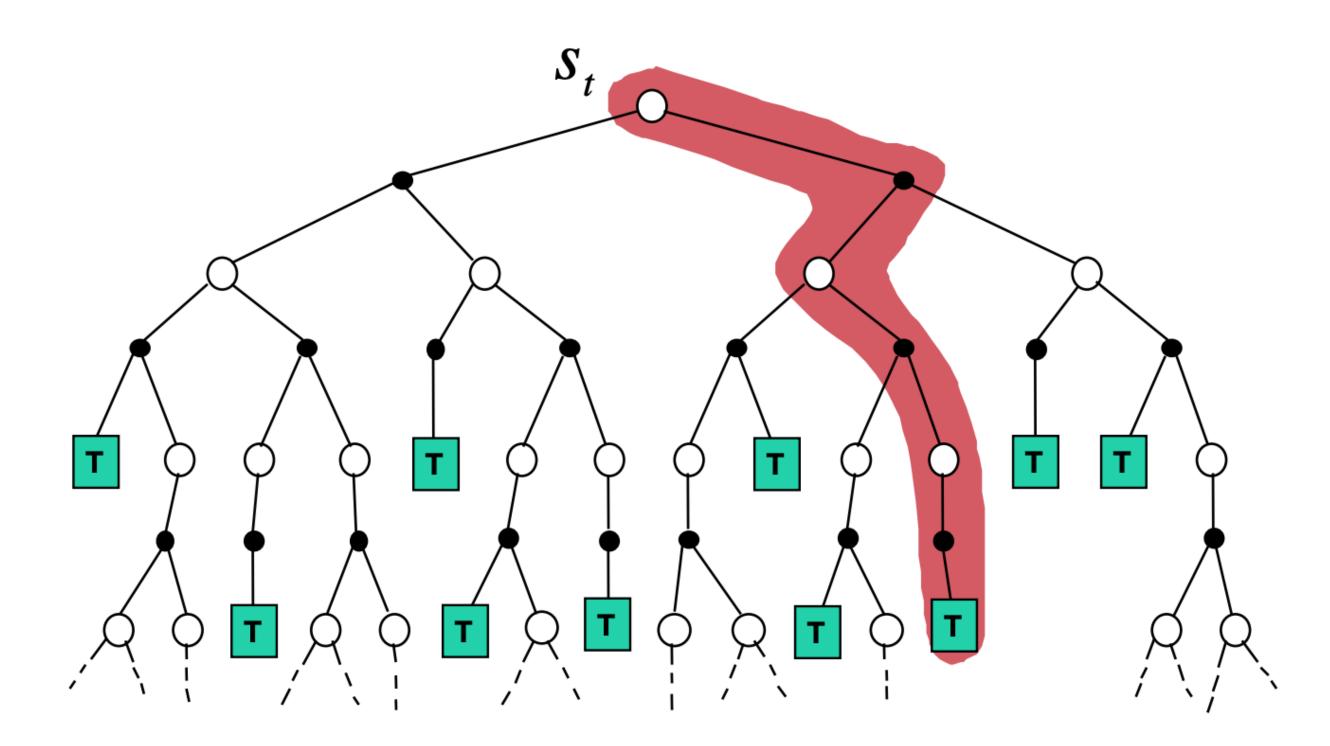
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• Model-based RL can learn very fast changes in the transition or reward distributions:

$$\Delta r(s_t,a_t,s_{t+1}) = lpha \left( r_{t+1} - r( 
ight) 
ight)$$

$$\Delta p(s'|s_t,a_t) = lpha \left( \mathbb{I}(s_{t+1}=s') 
ight)$$
 -

• But selecting an action requires planning in the tree of possibilities (slow).



$$egin{array}{l} & s_t, a_t, s_{t+1})) \ & - p(s'|s_t, a_t)) \end{array}$$

# **Model-based vs. Model-free**

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• Relative advantages of MF and MB methods:

	Inference speed	Sample complexity	Optimality	Flexibility
Model-free	fast	high	yes	no
Model-based	slow	low	as good as the model	yes

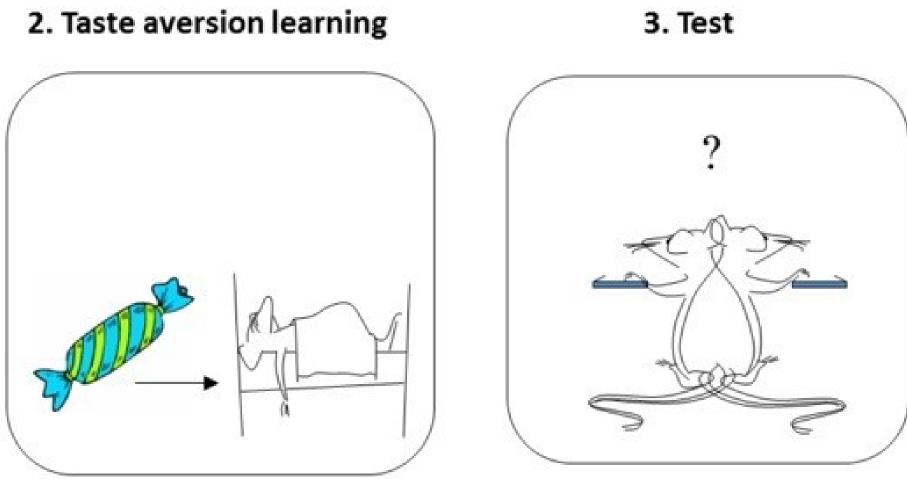
• A trade-off would be nice... Most MB models in the deep RL literature are hybrid MB/MF models anyway.

# **Outcome devaluation**

- Two forms of behavior are observed in the animal psychology literature:
- 1. **Goal-directed** behavior learns Stimulus  $\rightarrow$  Response  $\rightarrow$  Outcome associations.
- 2. Habits are developed by overtraining Stimulus  $\rightarrow$  Response associations.
- The main difference is that habits are not influenced by **outcome devaluation**, i.e. when rewards lose their value.



1. Instrumental Learning

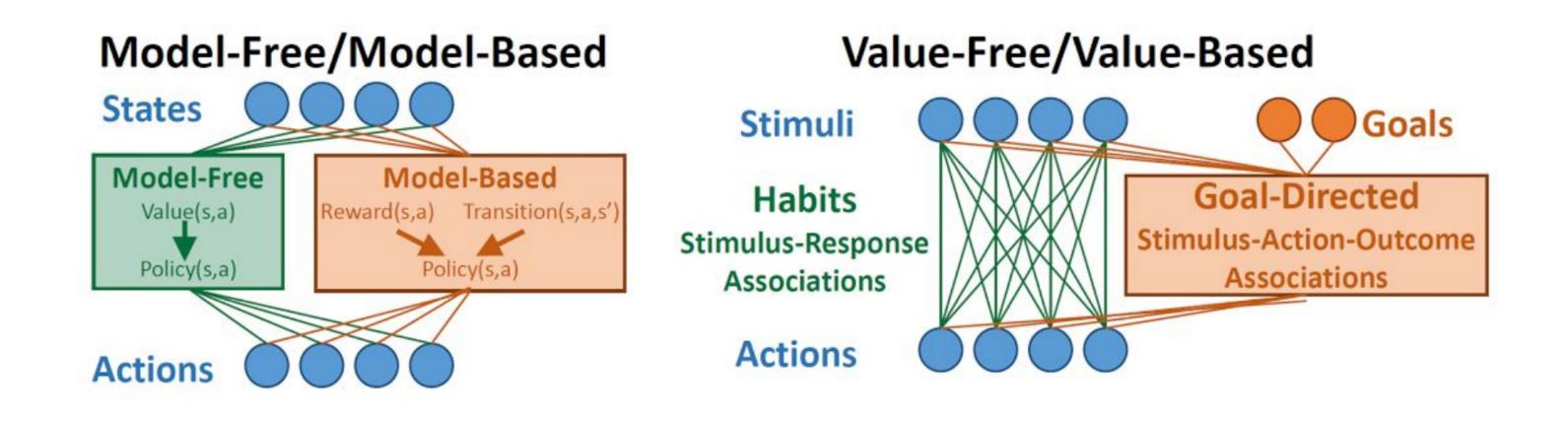


Source: Bernard W. Balleine

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# Goal-directed / habits = MB / MF?

• The classical theory assigns MF to habits and MB to goal-directed, mostly because their sensitivity to outcome devaluation.



- The open question is the arbitration mechanism between these two segregated process: who takes control?
- Recent work suggests both systems are largely overlapping.

### References

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Doll, B. B., Simon, D. A., and Daw, N. D. (2012). The ubiquity of model-based reinforcement learning. Current Opinion in Neurobiology 22, 1075–1081. doi:10.1016/j.conb.2012.08.003.

Miller, K., Ludvig, E. A., Pezzulo, G., and Shenhav, A. (2018). "Re-aligning models of habitual and goal-directed decision-making," in Goal-Directed Decision Making: Computations and Neural Circuits, eds. A. Bornstein, R. W. Morris, and A. Shenhav (Academic Press)

# 2 - Successor representations

• Successor representations (SR) have been introduced to combine MF and MB properties. Let's split the definition of the value of a state:

$$V^{\pi}(s) = \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, r_{t+k+1} | s_t = s]$$

$$= \mathbb{E}_{\pi} \left[ \sum_{k=0}^{\gamma} \gamma^{k} r_{t+k+1} | s_{t} = s \right]$$

$$= \mathbb{E}_{\pi} \left[ \begin{bmatrix} 1 \\ \gamma \\ \gamma^{2} \\ \cdots \\ \gamma^{\infty} \end{bmatrix} \times \begin{bmatrix} \mathbb{I}(s_{t}) \\ \mathbb{I}(s_{t+1}) \\ \mathbb{I}(s_{t+2}) \\ \cdots \\ \mathbb{I}(s_{\infty}) \end{bmatrix} \times \begin{bmatrix} r_{t+1} \\ r_{t+2} \\ r_{t+3} \\ \cdots \\ r_{t+\infty} \end{bmatrix} | s_{t} = s ]$$

$$(3)$$

where  $\mathbb{I}(s_t)$  is 1 when the agent is in  $s_t$  at time t, 0 otherwise.

- The left part corresponds to the **transition dynamics**: which states will be visited by the policy, discounted by  $\gamma$ .
- The right part corresponds to the **immediate reward** in each visited state.
- Couldn't we learn the transition dynamics and the reward distribution separately in a model-free manner?

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• SR rewrites the value of a state into an expected discounted future state occupancy  $M^{\pi}(s,s')$  and an expected immediate reward r(s') by summing over all possible states s' of the MDP:

$$V^{\pi}(s) = \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, r_{t+k+1} | s_t = s]$$

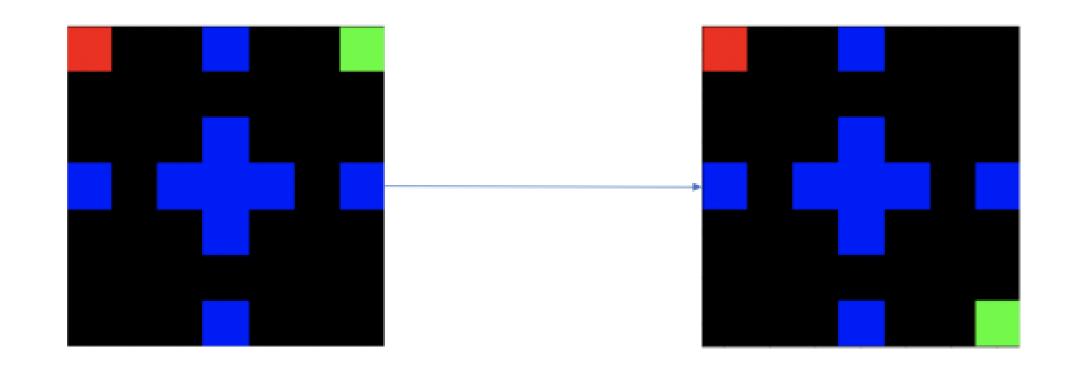
$$=\sum_{s'\in\mathcal{S}}\mathbb{E}_{\pi}[\sum_{k=0}^{\infty}\gamma^{k}\,\mathbb{I}(s_{t+k}=s') imes r)$$

$$pprox \sum_{s'\in\mathcal{S}}\mathbb{E}_{\pi}[\sum_{k=0}^{\infty}\gamma^{k}\,\mathbb{I}(s_{t+k}=s')|s_{t}>$$

$$pprox \sum_{s'\in\mathcal{S}} M^{\pi}(s,s') imes r(s')$$

(4)(5) $r_{t+k+1}|s_t=s|$ (6)(7) $|s| = s] imes \mathbb{E}[r_{t+1}|s_t = s']$ (8)(9)(10)

- The underlying assumption is that the world dynamics are independent from the reward function (which does not depend on the policy).
- This allows to re-use knowledge about world dynamics in other contexts (e.g. a new reward function in the same environment): transfer learning.



Source: https://awjuliani.medium.com/the-present-in-terms-of-the-future-successor-representations-in-reinforcement-learning-316b78c5fa3

- What matters is the states that you will visit and how interesting they are, not the order in which you visit them.
- Knowing that being in the mensa will eventually get you some food is enough to know that being in the mensa is a good state: you do not need to remember which exact sequence of transitions will put food in your mouth.

- SR algorithms must estimate two quantities:
  - 1. The **expected immediate reward** received after each state:

$$r(s) = \mathbb{E}[r_{t+1}|s_t = s]$$

2. The **expected discounted future state occupancy** (the **SR** itself):

$$M^{\pi}(s,s') = \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k}=s') | s_t=s]$$

• The value of a state *s* is then computed with:

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$$V^{\pi}(s) = \sum_{s' \in \mathcal{S}} M(s,s') imes$$

what allows to infer the policy (e.g. using an actor-critic architecture).

• The immediate reward for a state can be estimated very quickly and flexibly after receiving each reward:

$$\Delta \, r(s_t) = lpha \, (r_{t+1} - r(s_t))$$

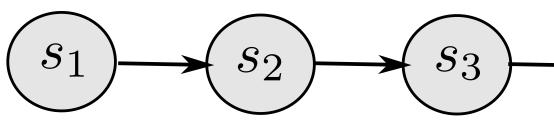
imes r(s') .

 $(s_t))$ 

# **SR and transition matrix**

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• Imagine a very simple MDP with 4 states and a single deterministic action:

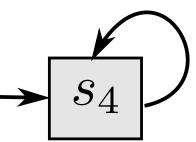


• The transition matrix  $\mathcal{P}^{\pi}$  depicts the possible (s,s') transitions:

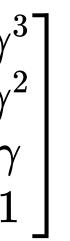
$$\mathcal{P}^{\pi} = egin{bmatrix} 0 & 1 & 0 & 0 \ 0 & 0 & 1 & 0 \ 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 1 \ \end{pmatrix}$$

• The SR matrix M also represents the future transitions discounted by  $\gamma$ :

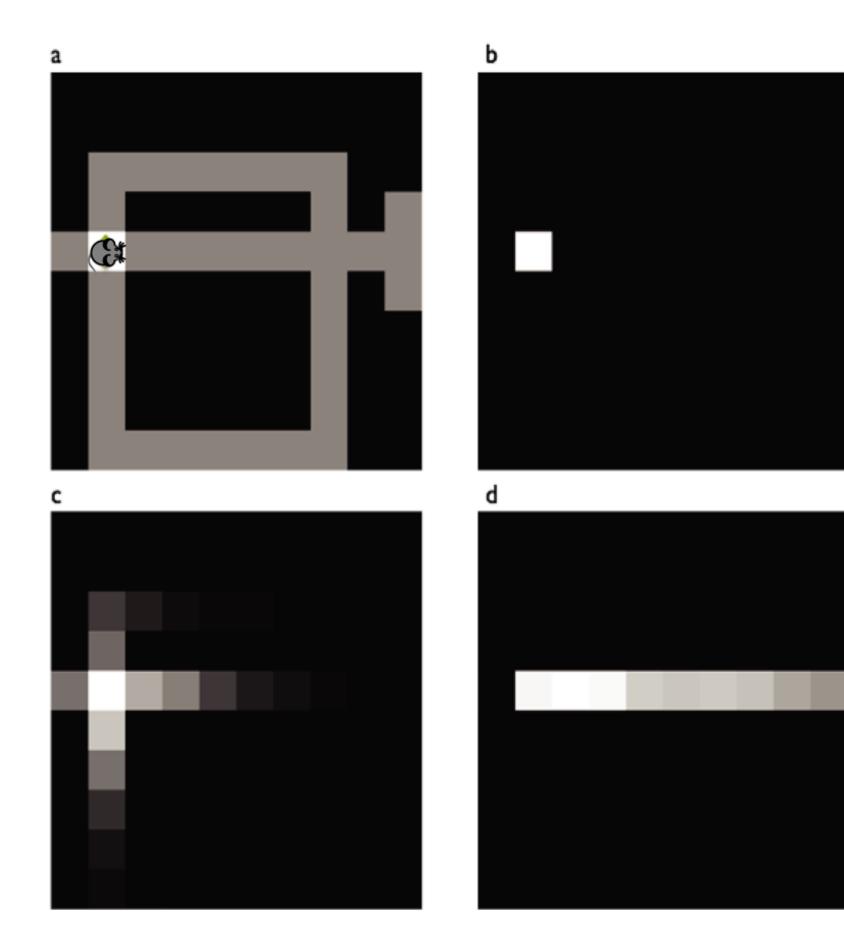
$$M = egin{bmatrix} 1 & \gamma & \gamma^2 & \gamma \ 0 & 1 & \gamma & \gamma \ 0 & 0 & 1 & \gamma \ 0 & 0 & 0 & 1 & \gamma \ 0 & 0 & 0 & 1 & \gamma \ \end{pmatrix}$$







## SR matrix in a Tolman's maze



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- policy.
- - A random agent will map the local neighborhood (c).
  - (d).
- how.

• The SR represents whether a state can be reached soon from the current state (b) using the current

• The SR depends on the policy:

A goal-directed agent will have SR representations that follow the optimal path

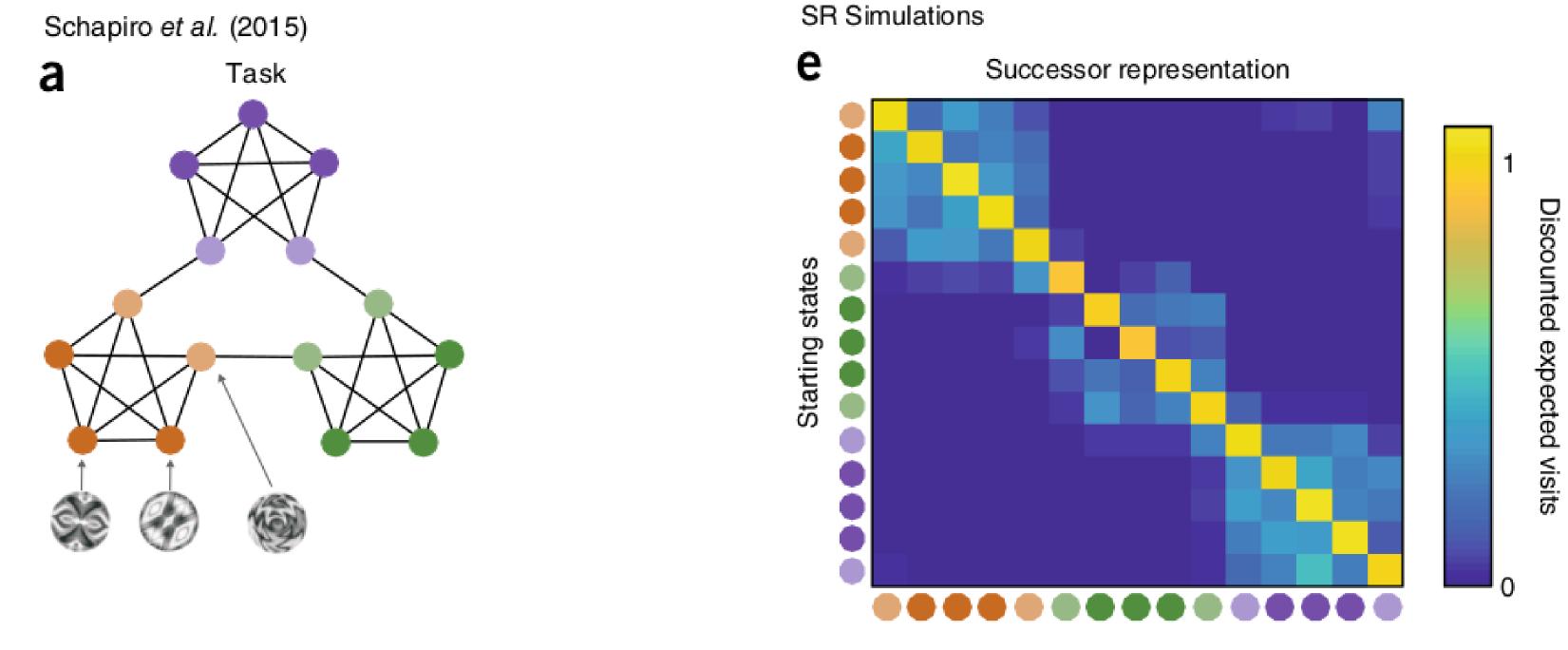
• It is therefore different from the transition matrix, as it depends on behavior and rewards.

• The exact dynamics are lost compared to MB: it only represents whether a state is reachable, not

# **Example of a SR matrix**

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• The SR matrix reflects the proximity between states depending on the transitions and the policy. it does not have to be a spatial relationship.



Visited states

# Learning the SR

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• How can we learn the SR matrix for all pairs of states?

$$M^{\pi}(s,s') = \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k}=s') | s_t=s]$$

• We first notice that the SR obeys a recursive Bellman-like equation:

$$egin{aligned} M^{\pi}(s,s') &= \mathbb{I}(s_t=s') + \mathbb{E}_{\pi}[\sum_{k=1}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k}=s') | s_t=s] \ &= \mathbb{I}(s_t=s') + \gamma \, \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k+1}=s') | s_t=s] \ &= \mathbb{I}(s_t=s') + \gamma \, \mathbb{E}_{s_{t+1}\sim \mathcal{P}^{\pi}(s'|s)}[\mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k}=s') | s_{t+1}=s]] \ &= \mathbb{I}(s_t=s') + \gamma \, \mathbb{E}_{s_{t+1}\sim \mathcal{P}^{\pi}(s'|s)}[M^{\pi}(s_{t+1},s')] \end{aligned}$$

• This is reminiscent of TDM: the remaining distance to the goal is 0 if I am already at the goal, or gamma the distance from the next state to the goal.

## **Model-based SR**

• Bellman-like SR:

$$M^{\pi}(s,s') = \mathbb{I}(s_t=s') + \gamma \, \mathbb{E}_{s_{t+1}\sim \mathcal{P}^{\pi}(s'|s)}[M^{\pi}(s_{t+1},s')]$$

• If we know the transition matrix for a fixed policy  $\pi$ :

$$\mathcal{P}^{\pi}(s,s') = \sum_a \pi(s,a) \, p(s'|s,a)$$

we can obtain the SR directly with matrix inversion as we did in **dynamic programming**:

$$M^{\pi} = I + \gamma \, \mathcal{P}^{\pi} imes N$$

so that:

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$$M^{\pi} = (I - \gamma \, \mathcal{P}^{\pi})^{-}$$

• This DP approach is called model-based SR (MB-SR) as it necessitates to know the environment dynamics.

 $M^{\pi}$ 

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### **Model-free SR**

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• If we do not know the transition probabilities, we simply sample a single  $s_t, s_{t+1}$  transition:

$$M^{\pi}(s_t,s') pprox \mathbb{I}(s_t=s') + \gamma\,M^{\pi}(s_{t+1},s')$$

• We can define a **sensory prediction error** (SPE):

$$\delta^{ ext{SR}}_t = \mathbb{I}(s_t = s') + \gamma \, M^{\pi}(s_{t+1},s') - M(s_t,s')$$

that is used to update an estimate of the SR:

$$\Delta M^{\pi}(s_t,s') = lpha \, \delta^{\mathrm{S}}_t$$

• This is **SR-TD**, using a SPE instead of RPE, which learns only from transitions but ignores rewards.

### SR

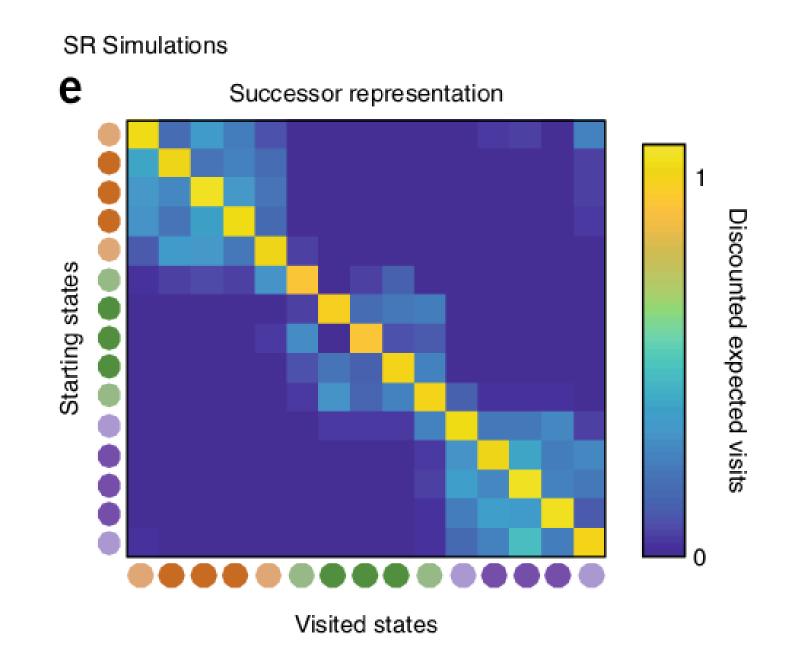
# **The sensory prediction error - SPE**

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• The SPE has to be applied on ALL successor states s' after a transition  $(s_t, s_{t+1})$ :

$$M^{\pi}(s_t,\mathbf{s'}) = M^{\pi}(s_t,\mathbf{s'}) + lpha\left(\mathbb{I}(s_t=\mathbf{s'}) + \gamma \, M^{\pi}(s_{t+1},\mathbf{s'}) - M(s_t,\mathbf{s'})
ight)$$

- Contrary to the RPE, the SPE is a **vector** of prediction errors, used to update one row of the SR matrix.
- The SPE tells how **surprising** a transition  $s_t \rightarrow s_{t+1}$  is for the SR.



## **Successor representations**

• The SR matrix represents the **expected discounted future state occupancy**:

$$M^{\pi}(s,s') = \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k}=s') | s_t=s]$$

• It can be learned using a TD-like SPE from single transitions:

$$M^{\pi}(s_t,\mathbf{s'}) = M^{\pi}(s_t,\mathbf{s'}) + lpha\left(\mathbb{I}(s_t=\mathbf{s'}) + \gamma\,M^{\pi}(s_{t+1},\mathbf{s'}) - M(s_t,\mathbf{s'})
ight)$$

• The immediate reward in each state can be learned **independently from the policy**:

$$\Delta \, r(s_t) = lpha \left( r_{t+1} - r(s_t) 
ight)$$

• The value  $V^{\pi}(s)$  of a state is obtained by summing of all successor states:

$$V^{\pi}(s) = \sum_{s' \in \mathcal{S}} M(s,s') imes$$

• This critic can be used to train an **actor**  $\pi_{\theta}$  using regular TD learning (e.g. A3C).

### **Successor representation of actions**

• Note that it is straightforward to extend the idea of SR to state-action pairs:

$$M^{\pi}(s,a,s') = \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k \, \mathbb{I}(s_{t+k}=s') | s_t=s, a_t=a]$$

allowing to estimate Q-values:

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$$Q^{\pi}(s,a) = \sum_{s' \in \mathcal{S}} M(s,a,s') imes r(s')$$

using SARSA or Q-learning-like SPEs:

$$\delta^{ ext{SR}}_t = \mathbb{I}(s_t = s') + \gamma \, M^{\pi}(s_{t+1}, a_{t+1},$$

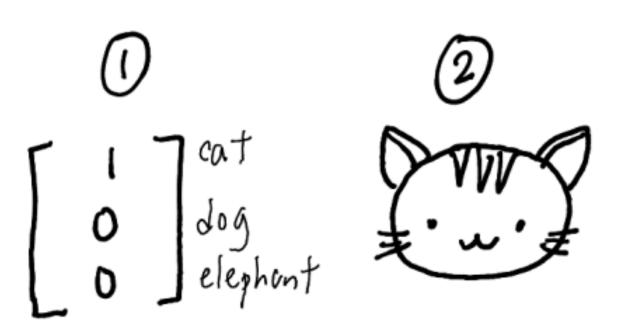
depending on the choice of the next action  $a_{t+1}$  (on- or off-policy

$$(s')-M(s_t,a_t,s')$$

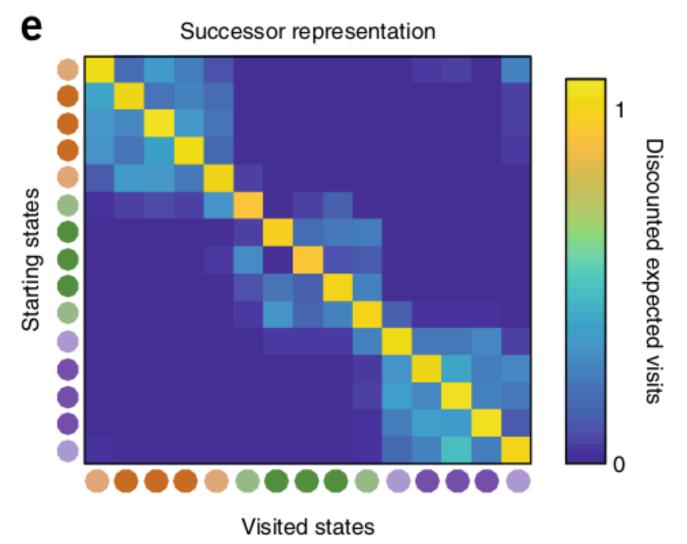
- The SR matrix associates each state to all others ( N imes Nmatrix):
  - curse of dimensionality.

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- only possible for discrete state spaces.
- A better idea is to describe each state *s* by a feature vector  $\phi(s) = [\phi_i(s)]_{i=1}^d$  with less dimensions than the number of states.
- This feature vector can be constructed (see the lecture on function approximation) or learned by an autoencoder (latent representation).



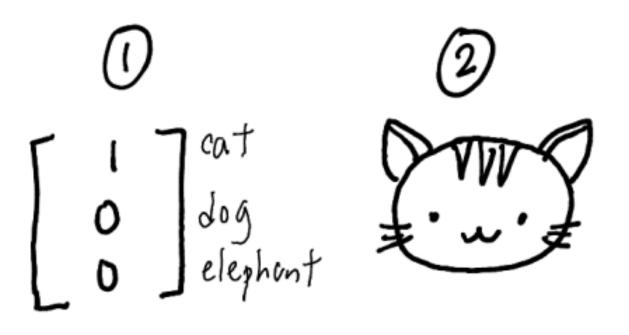
Source: http://www.jessicayung.com/the-successor-representation-1-generalising-between-states/



### SR Simulations

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• The successor feature representation (SFR) represents the discounted probability of observing a feature  $\phi_i$  after being in s.



Source: http://www.jessicayung.com/the-successor-representation-1-generalising-between-states/

• Instead of predicting when the agent will see a cat after being in the current state s, the SFR predicts when it will see eyes, ears or whiskers independently:

$$M^\pi_j(s) = M^\pi(s,\phi_j) = \mathbb{E}_\pi[\sum_{k=0}^\infty \gamma^k \, \mathbb{I}(\phi_j(s_{t+k})) | s_t = s, a_t = a]$$

• Linear SFR (Gehring, 2015) supposes that it can be linearly approximated from the features of the current state:

• The value of a state is now defined as the sum over successor features of their immediate reward discounted by the SFR:

$$V^{\pi}(s) = \sum_{j=1}^d M_j^{\pi}(s) \, r(\phi_j) = \sum_{j=1}^d r(\phi_j) \, \sum_{i=1}^d m_{i,j} \, \phi_i(s)$$

- The SFR matrix  $M^{\pi}=[m_{i,j}]_{i,j}$  associates each feature  $\phi_i$  of the current state to all successor features  $\phi_j$ .
  - Knowing that I see a kitchen door in the current state, how likely will I see a food outcome in the near future?
- Each successor feature  $\phi_j$  is associated to an expected immediate reward  $r(\phi_j)$ .
  - A good state is a state where food features (high  $r(\phi_j)$ ) are likely to happen soon (high  $m_{i,j}$ ).
- In matrix-vector form:

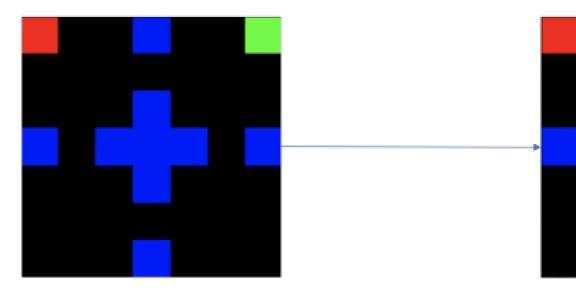
$$V^{\pi}(s) = \mathbf{r}^T imes M^{\pi} imes s$$

$$\phi(s)$$

• Value of a state:

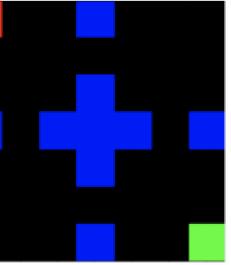
$$V^{\pi}(s) = \mathbf{r}^T imes M^{\pi} imes w$$

- The reward vector  ${f r}$  only depends on the features and can be learned independently from the policy, but can be made context-dependent:
  - Food features can be made more important when the agent is hungry, less when thirsty.
- **Transfer learning** becomes possible in the same environment:
  - Different goals (searching for food or water, going to place A or B) only require different reward vectors.
  - The dynamics of the environment are stored in the SFR.



Source: https://awjuliani.medium.com/the-present-in-terms-of-the-future-successor-representations-in-reinforcement-learning-316b78c5fa3

$$\phi(s)$$



• How can we learn the SFR matrix  $M^{\pi}$ ?

$$V^{\pi}(s) = \mathbf{r}^T imes M^{\pi} imes \phi(s)$$

• We only need to use the sensory prediction error for a transition between the feature vectors  $\phi(s_t)$  and  $\phi(s_{t+1})$ :

$$\delta^{
m SFR}_t = \phi(s_t) + \gamma\,M^\pi imes\phi(s_{t+1}) - M^\pi imes\phi(s_t)$$

and use it to update the whole matrix:

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$$\Delta M^{\pi} = \delta_t^{
m SFR} imes \phi(s_t)$$

• However, this linear approximation scheme only works for **fixed** feature representation  $\phi(s)$ . We need to go deeper...

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### **Deep Successor Reinforcement Learning**

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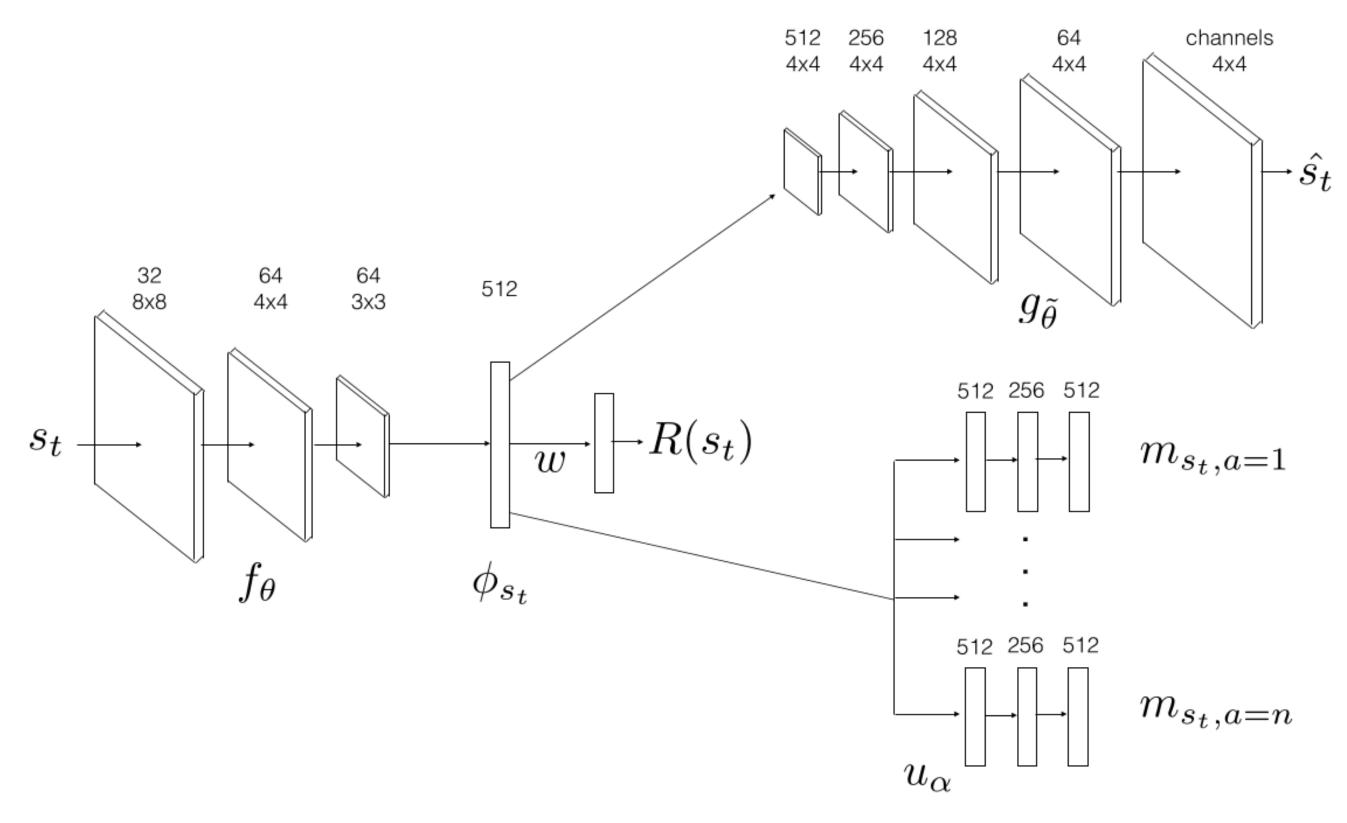


Figure 1: Model Architecture: DSR consists of: (1) feature branch  $f_{\theta}$  (CNN) which takes in raw images and computes the features  $\phi_{s_t}$ , (2) successor branch  $u_{\alpha}$  which computes the SR  $m_{s_t,a}$  for each possible action  $a \in A$ , (3) a deep convolutional decoder which produces the input reconstruction  $\hat{s}_t$  and (4) a linear regressor to predict instantaneous rewards at  $s_t$ . The Q-value function can be estimated by taking the inner-product of the SR with reward weights:  $Q^{\pi}(s, a) \approx m_{sa} \cdot \mathbf{w}$ .

- Each state  $s_t$  is represented by a D-dimensional (D=512) vector  $\phi(s_t)=f_ heta(s_t)$  which is the output of an encoder.
- A decoder  $g_{\hat{ heta}}$  is used to provide a reconstruction loss, so  $\phi(s_t)$  is a latent representation of an autoencoder:

$$\mathcal{L}_{ ext{reconstruction}}( heta, \hat{ heta}) = \mathbb{E}[(g_{\hat{ heta}}(\phi(s_t)) - s_t)^2]$$

• The immediate reward  $R(s_t)$  is linearly predicted from the feature vector  $\phi(s_t)$  using a reward vector  $\mathbf{w}$ .

$$R(s_t) = \phi(s_t)^T imes \mathbf{v}$$

$$\mathcal{L}_{ ext{reward}}(\mathbf{w}, heta) = \mathbb{E}[(r_{t+1} - \phi(s_t)^T imes \mathbf{w})^2]$$

- The reconstruction loss is important, otherwise the latent representation  $\phi(s_t)$  would be too rewardoriented and would not generalize.
- The reward function is learned on a single task, but it can fine-tuned on another task, with all other weights frozen.

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• For each action a, a NN  $u_{lpha}$  predicts the future feature occupancy M(s,s',a) for the current state:

$$m_{s_t a} = u_lpha(s_t, a)$$
 .

• The Q-value of an action is simply the dot product between the SR of an action and the reward vector  $\mathbf{w}$ :

$$Q(s_t,a) = \mathbf{w}^T imes m_s$$

• The selected action is  $\epsilon$ -greedily selected around the greedy action:

$$a_t = rg\max_a Q(s_t,$$

• The SR of each action is learned using the Q-learning-like SPE (with fixed  $\theta$  and a target network  $u_{\alpha'}$ ):

$$\mathcal{L}^{ ext{SPE}}(lpha) = \mathbb{E}[\sum_{a} (\phi(s_t) + \gamma \, \max_{a'} u_{lpha'}(s_t)]$$

• The compound loss is used to train the complete network end-to-end **off-policy** using a replay buffer (DQN-like).

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$$(A \hat{A} \mathbf{x} \mathbf{x} \mathbf{x} \mathbf{y}) \stackrel{\text{Kulkarni et al. (2016) Deep Successor Reinforcement Learning. arXiv:1606.02396}{(A A) \perp (\mathbf{x} \mathbf{x} \mathbf{x} A) \perp (\mathbf{x} \mathbf{x} \mathbf{y} A) \perp (\mathbf{x} \mathbf{x} \mathbf{y} A) \perp (\mathbf{x} \mathbf{y} \mathbf{y} A)$$

 $s_t a$ 

a)

 $[s_{t+1},a') - u_{lpha}(s_t,a))^2]$ 

**Algorithm 1** Learning algorithm for DSR

- 1: Initialize experience replay memory  $\mathcal{D}$ , parameters  $\{\theta, \alpha, \mathbf{w}, \hat{\theta}\}$  and exploration probability  $\epsilon = 1.$ 2: for i = 1 : #episodes do Initialize game and get start state description s3: while not terminal do 4:  $\phi_s = f_\theta(s)$ 5: With probability  $\epsilon$ , sample a random action a, otherwise choose  $\operatorname{argmax}_a u_\alpha(\phi_s, a) \cdot \mathbf{w}$ 6:
- Execute a and obtain next state s' and reward R(s') from environment 7:
- Store transition (s, a, R(s'), s') in  $\mathcal{D}$ 8:
- Randomly sample mini-batches from  ${\cal D}$ 9:
- Perform gradient descent on the loss  $L^r(\mathbf{w}, \theta) + L^a(\tilde{\theta}, \theta)$  with respect to  $\mathbf{w}, \theta$  and  $\tilde{\theta}$ . 10:
- Fix  $(\theta, \tilde{\theta}, \mathbf{w})$  and perform gradient descent on  $L^m(\alpha, \theta)$  with respect to  $\alpha$ . 11:
- $s \leftarrow s'$ 12:
- end while 13:
- Anneal exploration variable  $\epsilon$ 14:
- 15: **end for**

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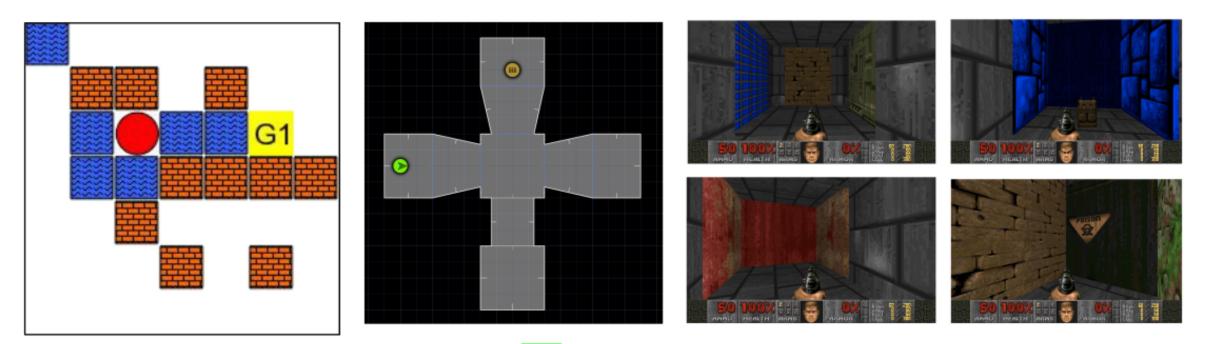


Figure 2: Environments: (left) MazeBase [37] map where the agent starts at an arbitrary location and needs to get to the goal state. The agent gets a penalty of -0.5 per-step, -1 to step on the water-block (blue) and +1 for reaching the goal state. The model observes raw pixel images during learning. (center) A *Doom* map using the VizDoom engine [13] where the agent starts in a room and has to get to another room to collect ammo (per-step penalty = -0.01, reward for reaching goal = +1). (right) Sample screen-shots of the agent exploring the 3D maze.

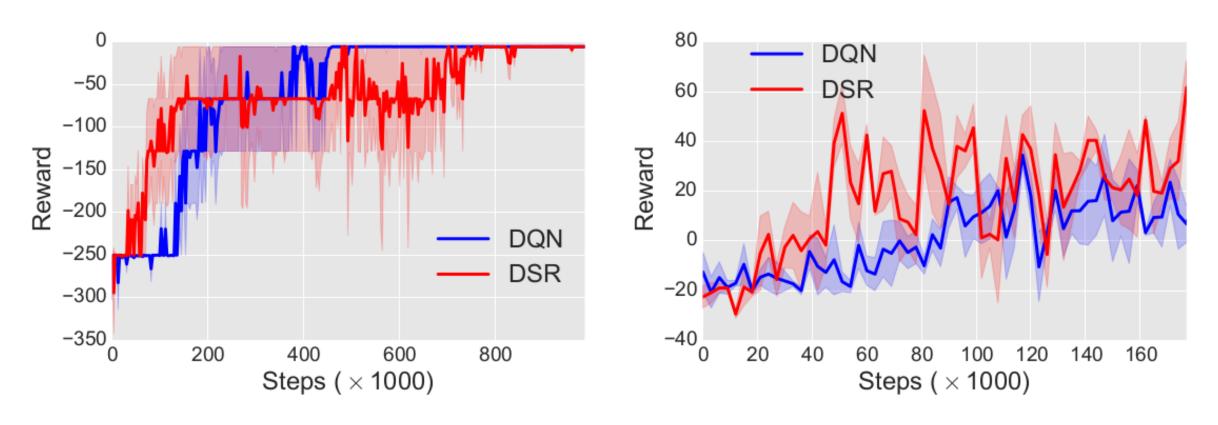
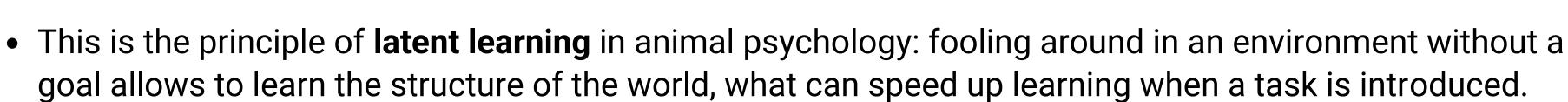


Figure 3: Average trajectory of the reward (left) over 100k steps for the grid-world maze. (right) over 180k steps for the Doom map over multiple runs.

- The interesting property is that you do not need rewards to learn:
  - A random agent can be used to learn the encoder and the SR, but w can be left untouched.
  - When rewards are introduced (or changed), only w has to be adapted, while DQN would have to re-learn all Q-values.

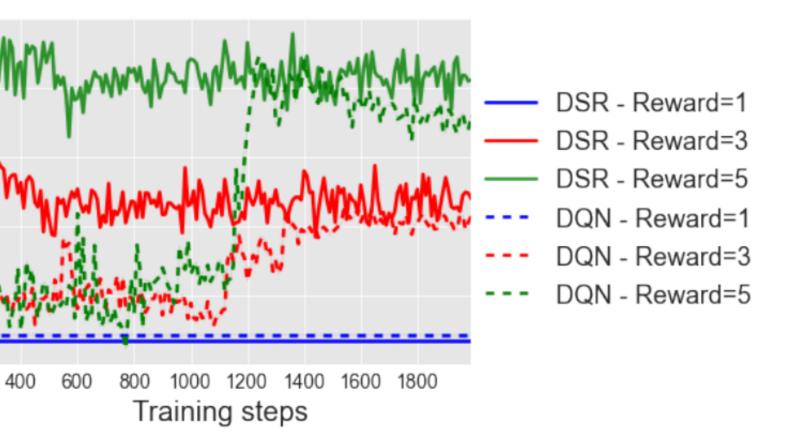
 $\equiv$ 



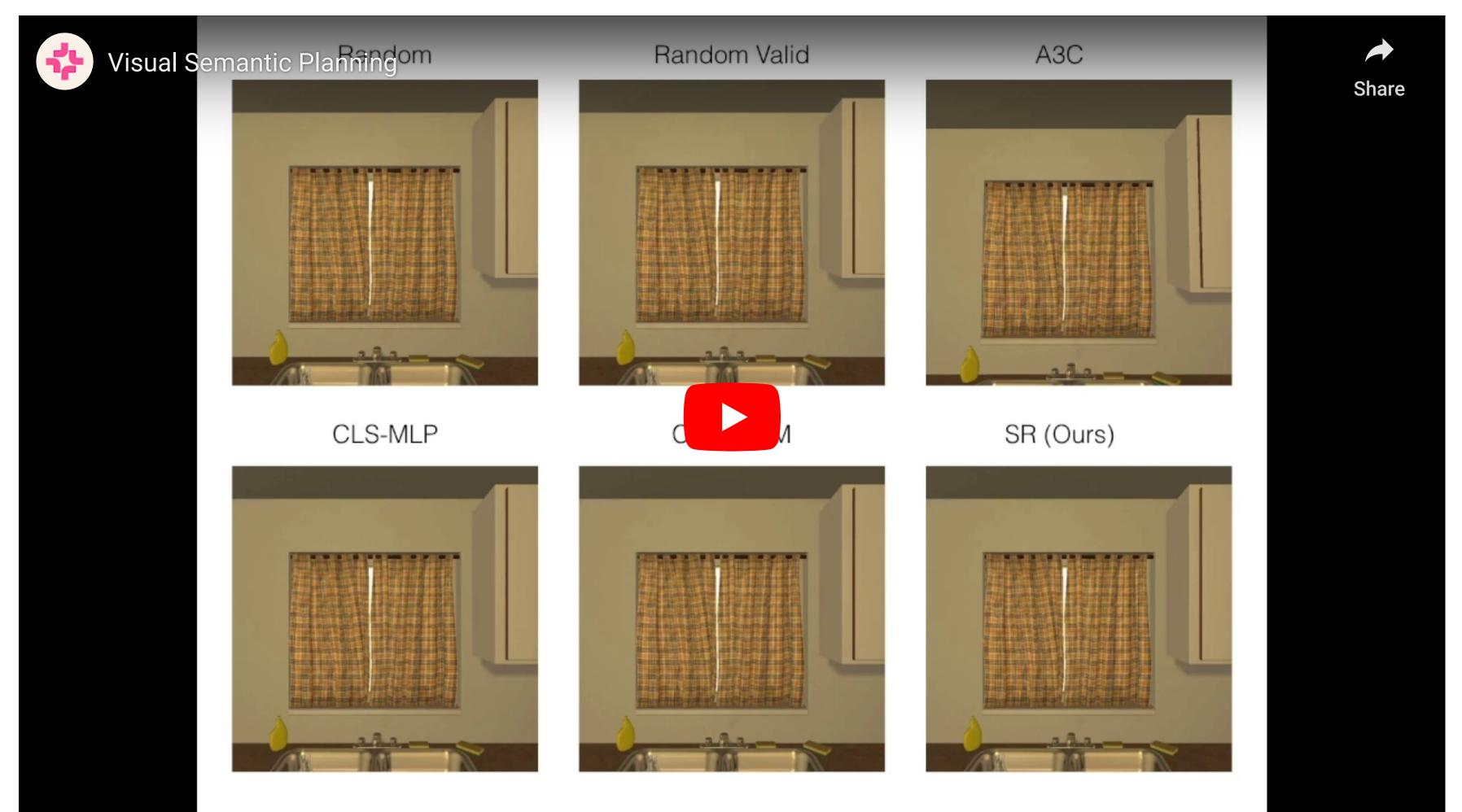
• The SR is a **cognitive map** of the environment: learning task-unspecific relationships.

Q-value at the origin

-3



# Visual Semantic Planning using Deep Successor Representations



# References

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